AIAA JOURNAL
Vol. 39, No. 8, August 2001

Inverse Sensitivity Method for Asymmetric Multivariable Nondefective Systems

L. F. Chen,* H. W. Song,† and W. L. Wang‡
Fudan University, 200433 Shanghai, People's Republic of China

An available parametric correction method for asymmetric multivariable nondefective systems is investigated. It will be proved that the eigensolutions associated with the repeated eigenvalues are not differentiable, although their partial derivatives exist. A calculable generalized Jacobian matrix is presented based on the first-order Taylor expansion of the repeated root eigenpairs as the case of symmetric systems, which ensures the feasibility of the inverse sensitivity method for parametric correction. The first-order directional derivatives of the repeated root eigenpairs are also found. Two numerical asymmetric examples are given to verify the validity of the method by use of simulated data.

Nomenclature

$\boldsymbol{D}_t$	=	h-order coefficient matrix
$\boldsymbol{e}_i$	=	ith natural base vector of n-dimensional parametric space
h	=	multiplicity of the eigenvalue
$\boldsymbol{I}_h$	=	h-order identity matrix
$\boldsymbol{J}_e$	=	generalized Jacobian matrix
$\boldsymbol{K}$	=	structural real stiffness matrix
$\boldsymbol{M}$	=	structural real positive definite mass matrix
$\boldsymbol{P}_t$	=	projection matrix
$\boldsymbol{p}$	=	vector of structural physical parameters of dimension n
$\boldsymbol{X}_t$	=	arbitrary $\boldsymbol{M}$-biorthogonalized eigenvector subset with respect to h-fold eigenvalue
$\boldsymbol{Y}_t$	=	complementary subset with respect to $\boldsymbol{X}_t$
$\boldsymbol{Z}_t$	=	$\boldsymbol{M}$-biorthogonalized eigenvector subset with respect to h-fold eigenvalue whose partial derivatives and directional derivatives along a certain direction exist; so-called derivable eigenvector subset
$\bar{\boldsymbol{Z}}$	=	test mode matrix of experimental model
Γ_t	=	transform matrix from $\boldsymbol{X}_t$ to $\boldsymbol{Z}_t$
$\Theta_t(\boldsymbol{\tau}, \boldsymbol{e}_i)$	=	transform matrix from $\boldsymbol{Z}_t(\boldsymbol{\tau})$ to $\boldsymbol{Z}_t(\boldsymbol{e}_i)$
$\boldsymbol{\kappa}$	=	frequency shift stiffness matrix, $\boldsymbol{K} - \lambda\boldsymbol{M}$
Λ	=	diagonal matrix of h-fold eigenvalue, $\lambda\boldsymbol{I}_h$
$\bar{\Lambda}$	=	diagonal matrix of test eigenvalues of experimental model
λ	=	h-fold eigenvalue
$\boldsymbol{\tau}$	=	unit vector of an arbitrary direction in n-dimensional parametric space

Subscripts

t	=	R (right) or L (left)
$,\tau$	=	$\partial(\,)/\partial\boldsymbol{\tau}\|_{p=p_0}$
$,\tau\tau$	=	$\partial^2(\,)/\partial\boldsymbol{\tau}^2\|_{p=p_0}$

Superscripts

T	=	transpose of matrix ( ) (not Hermitian transpose)
$+$	=	Moore–Penrose inverse of matrix ( )

Received 7 August 2000; revision received 23 January 2001; accepted for publication 27 January 2001.

*Ph.D. Candidate, Department of Mechanics and Engineering Science, 220 Handan Road.
†Associate Professor, Department of Mechanics and Engineering Science, 220 Handan Road.
‡Professor, Department of Mechanics and Engineering Science, 220 Handan Road.

Introduction

THE sensitivity and inverse sensitivity methods have been shown to be efficient and available in the fields of structural control, model modification, and design optimization because of their ability to provide analytical representations that are well correlated to experimental data and that preserve the physical significance of the models. Many authors have been researching this area in the past 30 years. In particular, Nelson,[1] Dailey,[2] and Mills-Curran[3] presented a series of algorithms for computing the eigensolutions sensitivity, which provided guidance in the inverse dynamic design; Prells and Friswell[4] discussed the problem of calculating partial derivatives of repeated root eigensolutions; Pešek[5] first extended an inverse sensitivity method to systems with repeated eigenvalues; and Song[6] gave an improved inverse sensitivity method. However, most studies have been limited to symmetric systems.

As modern control devices, asymmetric vibration systems will be encountered frequently in the context of gyroscopic and follower forces or calculating the aerodynamic derivatives of bridges. Several authors have considered asymmetric systems in the past two decades. Caughey and Ma[7] discussed the complex modes and solvability of asymmetric systems. Tang and Wang[8] and Wang[9] considered the sensitivity calculation for nondefective systems for the case of a single parameter. An organized course of the inverse sensitivity method for multivariable asymmetric systems has not been reported in the open literature. This paper deals with the case of asymmetric multivariable nondefective systems and proposes a new procedure of parametric correction based on the first-order Taylor series expansion of these systems. Because a defective system does not have a complete eigenvectors set, its sensitivity expressions will be more complex,[10] and more research should be carried out for the inverse sensitivity analysis of defective systems.

For asymmetric systems, the eigenvectors from the eigenequation and the biorthonormality condition are uncertain for a nonzero constant multiplier, and therefore, confusion exists between numerical results and experimental data because of normalization conditions. This paper will adopt the normalization conditions presented by Tang and Wang[8] in terms of modes of analytical models compared with the corresponding measured ones. Song[6] considered the differentiability of the repeated root eigenpairs for multivariable symmetric systems and concluded that it is not differentiable in general. The case for asymmetric systems is quite similar. The first section of this paper discusses the uniqueness and differentiability of the repeated root eigensolutions for multivariable asymmetric systems and also finds their first directional derivatives.

It is well known that parametric correction based on the first-order Taylor series expansion is quite straightforward for simple eigenvalues. However, this complicates the construction of multiple eigenvalues due to the nondifferentiability of the repeated root eigenpairs. The difficulty results because the Jacobian matrix (sensitivity matrix) not only depends on design parameters $\boldsymbol{p} = \boldsymbol{p}_0$ but also relates to parameters change $\Delta\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p}_0$. Furthermore, the dependence on

$\Delta \boldsymbol{p}$ for the homogeneous solution part in the directional derivatives of repeated root eigenvectors is highly nonlinear. The relationship is so complex that the Jacobian matrix cannot be written out easily. Song[6] gave a calculable generalized Jacobian matrix, which is formed by using projection matrices,[11] to reject the homogeneous solution part that is the linear combination of the derivable eigenvectors through biorthogonal decomposition. The second section of this paper will generalize Song's[6] result to asymmetric systems. The method includes previously developed methods for symmetric systems as special cases.

In the third section, two numerical examples are given, for which the availability and the efficiency of the presented method will be discussed by using simulated data.

Uniqueness and Differentiability of Repeated Eigenvalues and Corresponding Eigenvectors

Consider the following generalized eigenvalue problem:

$$\boldsymbol{K}(\boldsymbol{p})\Phi_R(\boldsymbol{p}) = \boldsymbol{M}(\boldsymbol{p})\Phi_R(\boldsymbol{p})\boldsymbol{\Omega}(\boldsymbol{p}) \tag{1}$$

$$\boldsymbol{K}(\boldsymbol{p})^T\Phi_L(\boldsymbol{p}) = \boldsymbol{M}(\boldsymbol{p})^T\Phi_L(\boldsymbol{p})\boldsymbol{\Omega}(\boldsymbol{p}) \tag{2}$$

with the mass biorthonormalization condition

$$\Phi_L(\boldsymbol{p})^T\boldsymbol{M}(\boldsymbol{p})\Phi_R(\boldsymbol{p}) = \boldsymbol{I}_N \tag{3}$$

where stiffness matrix $\boldsymbol{K}(\boldsymbol{p})$ and mass matrix $\boldsymbol{M}(\boldsymbol{p})$ for the N-degree-of-freedom discrete vibration system $[\boldsymbol{K}(\boldsymbol{p}), \boldsymbol{M}(\boldsymbol{p})]$ are single-valued limitary functions and have continuous partial derivatives of all orders with respect to multiple design parameters $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)^T$. $\boldsymbol{K}(\boldsymbol{p})$ and $\boldsymbol{M}(\boldsymbol{p})$ are real matrices, and $\boldsymbol{M}(\boldsymbol{p})$ is positive definite.

Obviously, the right eigenvectors $\Phi_R(\boldsymbol{p})$ and left eigenvector $\Phi_L(\boldsymbol{p})$ given by Eqs. (1–3) are uncertain to the extent of a nonzero constant multiplier. A new normalization condition, mentioned by Tang and Wang[8] and Wang,[9] should be imposed to result in unique eigenvectors, that is, every column $\varphi_{Ri}(\boldsymbol{p})$, $i = 1, 2, \ldots, N$, in the right eigenvector matrix $\Phi_R(\boldsymbol{p})$ is normalized according to mass matrix $\boldsymbol{M}(\boldsymbol{p})$:

$$\varphi_{Ri}(\boldsymbol{p})^T\boldsymbol{M}(\boldsymbol{p})\varphi_{Ri}(\boldsymbol{p}) = 1 \qquad (i = 1, 2, \ldots, N) \tag{4}$$

However, as pointed out by Murthy and Haftka,[12] confusion exists in previous literature regarding this point. For a non-self-adjoint system, condition (4) may fail even if $\boldsymbol{M}(\boldsymbol{p})$ is real, for example, if $\boldsymbol{M}(\boldsymbol{p})$ is skew symmetric. At that time, an alternative condition may be adopted.[8] Because $\boldsymbol{M}(\boldsymbol{p})$ is positive definite in this text, condition (4) will be mostly valid.

Therefore, the biorthonormalization condition (3) and additional condition (4) will uniquely determine the right and left eigenvectors.

Let the complete eigensolution set of Eqs. (1–4) be partitioned by

$$\Phi_t(\boldsymbol{p}) = \{\boldsymbol{Z}_t(\boldsymbol{p}), \boldsymbol{Y}_t(\boldsymbol{p})\} \qquad (t = R, L) \tag{5}$$

$$\boldsymbol{\Omega}(\boldsymbol{p}) = \mathrm{diag}[\Lambda(\boldsymbol{p}), \Xi(\boldsymbol{p})] \tag{6}$$

Suppose that at $\boldsymbol{p} = \boldsymbol{p}_0$ one is given an arbitrary h-fold $(1 \le h \le N)$ repeated eigenvalue $\Lambda(\boldsymbol{p}) = \lambda \boldsymbol{I}_h$ (here the case of $h > 1$ is emphasized) with corresponding eigenvector subsets $\boldsymbol{Z}_t(\boldsymbol{p}) = \{z_{t1}, z_{t2}, \ldots, z_{th}\}$, $t = R$ and L, normalized by conditions (3) and (4). Obviously, eigenpairs $\Lambda(\boldsymbol{p})$ and $\boldsymbol{Z}_t(\boldsymbol{p})$, $t = R$ and L, are the solutions of the following generalized eigenvalue problem:

$$\boldsymbol{K}(\boldsymbol{p})\boldsymbol{Z}_R(\boldsymbol{p}) = \boldsymbol{M}(\boldsymbol{p})\boldsymbol{Z}_R(\boldsymbol{p})\Lambda(\boldsymbol{p}) \tag{7}$$

$$\boldsymbol{K}(\boldsymbol{p})^T\boldsymbol{Z}_L(\boldsymbol{p}) = \boldsymbol{M}(\boldsymbol{p})^T\boldsymbol{Z}_L(\boldsymbol{p})\Lambda(\boldsymbol{p}) \tag{8}$$

$$\boldsymbol{Z}_L(\boldsymbol{p})^T\boldsymbol{M}(\boldsymbol{p})\boldsymbol{Z}_R(\boldsymbol{p}) = \boldsymbol{I}_h \tag{9}$$

$$z_{Ri}(\boldsymbol{p})^T\boldsymbol{M}(\boldsymbol{p})z_{Ri}(\boldsymbol{p}) = 1 \qquad (i = 1, 2, \ldots, h) \tag{10}$$

and, furthermore, on the assumption that when $\boldsymbol{p} \to \boldsymbol{p}_0$

$$\boldsymbol{K}(\boldsymbol{p}) \to \boldsymbol{K}, \qquad \boldsymbol{M}(\boldsymbol{p}) \to \boldsymbol{M}$$

$$\boldsymbol{\Omega}(\boldsymbol{p}) \to \boldsymbol{\Omega} = (\Lambda, \Xi), \qquad \Phi_t(\boldsymbol{p}) \to \Phi_t = \{\boldsymbol{Z}_t, \boldsymbol{Y}_t\}$$

hereafter $(\boldsymbol{p}_0)$ will be omitted for variables evaluated at $\boldsymbol{p} = \boldsymbol{p}_0$, similarly,

$$\boldsymbol{Z}_{t,\tau}(\boldsymbol{p}) \to \boldsymbol{Z}_{t,\tau} \qquad \Lambda_{,\tau}(\boldsymbol{p}) \to \Lambda_{,\tau} \qquad (t = R, L)$$

where $\boldsymbol{F}_{,\tau}, \boldsymbol{F}_{,\tau\tau}$, etc., are the first, second, etc., directional derivatives of matrix $\boldsymbol{F}$ evaluated at $\boldsymbol{p} = \boldsymbol{p}_0$ along the direction

$$\boldsymbol{\tau} = \frac{(p_1, p_2, \ldots, p_n)^T}{\sqrt{\sum_{i=1}^{n}(p_i)^2}}$$

It was concluded previously[8] that both the right and left eigenvectors $\boldsymbol{X}_R$ and $\boldsymbol{X}_L$ corresponding to repeated eigenvalues are degenerate. Mathematically, for any $h \times h$ matrices Γ_R and Γ_L satisfying $\Gamma_L^T\Gamma_R = \boldsymbol{I}_h$, $\boldsymbol{X}_R\Gamma_R$ and $\boldsymbol{X}_L\Gamma_L$ will also be the right and left eigenvector subsets. If the derivatives of the repeated eigenvalues are all distinct, under proper normalization conditions, only specific:

$$\boldsymbol{Z}_R = \boldsymbol{X}_R\Gamma_R \tag{11a}$$

$$\boldsymbol{Z}_L = \boldsymbol{X}_L\Gamma_L \tag{11b}$$

corresponding to unique Γ_R and Γ_L are derivable with respect to a certain direction $\boldsymbol{\tau}$, which means the directional derivatives along direction $\boldsymbol{\tau}$ exist.

By differentiating Eq. (7) along direction τ, letting $\boldsymbol{p} \to \boldsymbol{p}_0$, and denoting

$$\boldsymbol{\kappa}_\tau = \boldsymbol{K}_{,\tau} - \lambda\boldsymbol{M}_{,\tau} \tag{12a}$$

$$\boldsymbol{\kappa}_{\tau\tau} = \boldsymbol{K}_{,\tau\tau} - \lambda\boldsymbol{M}_{,\tau\tau} \tag{12b}$$

we obtain

$$\boldsymbol{\kappa}_\tau\boldsymbol{Z}_R = \boldsymbol{M}\boldsymbol{Z}_R\Lambda_{,\tau} - (\boldsymbol{K} - \lambda\boldsymbol{M})\boldsymbol{Z}_{R,\tau} \tag{13}$$

Left-multiplying Eq. (13) by $\boldsymbol{Z}_L^T$ and using $\boldsymbol{Z}_L^T(\boldsymbol{K} - \lambda\boldsymbol{M}) = 0$ and condition (9), we find

$$\Lambda_{,\tau} = \boldsymbol{Z}_L^T\boldsymbol{\kappa}_\tau\boldsymbol{Z}_R \tag{14}$$

When condition (9), transformations (11a) and (11b), and Eq. (14) are noted and $\boldsymbol{A} = \boldsymbol{X}_L^T\boldsymbol{\kappa}_\tau\boldsymbol{X}_R$ is introduced, transform matrices Γ_R and Γ_L can be obtained from the following standard eigenvalue problem of matrix $\boldsymbol{A}$:

$$\boldsymbol{A}\Gamma_R = \Gamma_R\Lambda_{,\tau} \tag{15}$$

$$\boldsymbol{A}^T\Gamma_L = \Gamma_L\Lambda_{,\tau} \tag{16}$$

$$\Gamma_L^T\Gamma_R = \boldsymbol{I}_h \tag{17}$$

Therefore, we can uniquely determine the derivable eigenvectors $\boldsymbol{Z}_R$ and $\boldsymbol{Z}_L$ after getting a set of arbitrary eigenvectors $\boldsymbol{X}_R$ and $\boldsymbol{X}_L$ corresponding to repeated eigenvalues by imposing transformations (11a) and (11b). In this way, the derivatives of repeated eigenvalues can be evaluated by Eq. (14).

Let $\boldsymbol{\kappa} = \boldsymbol{K} - \lambda\boldsymbol{M}$, and note Eqs. (12a) and (12b); $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}_\tau$, and $\boldsymbol{\kappa}_{\tau\tau}$ are not directly derivatives of each other. Then Eq. (13) can be rewritten as

$$\boldsymbol{\kappa}\boldsymbol{Z}_{R,\tau} = \boldsymbol{M}\boldsymbol{Z}_R\Lambda_{,\tau} - \boldsymbol{\kappa}_\tau\boldsymbol{Z}_R \stackrel{\mathrm{def}}{=} \boldsymbol{F}_1 \tag{18}$$

Because the kernel of $\boldsymbol{\kappa}$ is spanned by the columns of $\boldsymbol{Z}_R$, the solution set of Eq. (18) is given by

$$\boldsymbol{Z}_{R,\tau} = \boldsymbol{V}_{R,\tau} + \boldsymbol{Z}_R\boldsymbol{D}_R \tag{19}$$

where $\boldsymbol{D}_R$ is an $h \times h$ coefficient matrix to be determined and $\boldsymbol{Z}_R\boldsymbol{D}_R$ represents the homogeneous solution part, which is the linear combination of the derivable eigenvectors. The particular solution part $\boldsymbol{V}_{R,\tau}$ in Eq. (18) can be rewritten as

$$\boldsymbol{V}_{R,\tau} = \boldsymbol{\kappa}^{\{1\}}\boldsymbol{F}_1 \tag{20}$$

where $\boldsymbol{\kappa}^{\{1\}} \in \{\boldsymbol{G} \mid \boldsymbol{\kappa}\boldsymbol{G}\boldsymbol{\kappa} = \boldsymbol{\kappa}, \forall\boldsymbol{\kappa} \in C^{N\times N}\}$ and $\boldsymbol{G}$ is referred to as the generalized 1-inverse set of matrix $\boldsymbol{\kappa}$. Notice that the generalized 1-inverses $\boldsymbol{\kappa}^{\{1\}}$ are not unique, nor is the preceding particular solution $\boldsymbol{V}_{R,\tau}$. In virtue of the right and left projection matrices[11]

$$\boldsymbol{P}_R \overset{\text{def}}{=} \boldsymbol{I}_N - \boldsymbol{M}\boldsymbol{X}_R\boldsymbol{X}_L^T = \boldsymbol{I}_N - \boldsymbol{M}\boldsymbol{Z}_R\boldsymbol{Z}_L^T \tag{21}$$

$$\boldsymbol{P}_L \overset{\text{def}}{=} \boldsymbol{I}_N - \boldsymbol{M}^T\boldsymbol{X}_L\boldsymbol{X}_R^T = \boldsymbol{I}_N - \boldsymbol{M}^T\boldsymbol{Z}_L\boldsymbol{Z}_R^T \tag{22}$$

it is easy to verify that[8,9]

$$\boldsymbol{P}_L^T\boldsymbol{\kappa}^{\{1\}}\boldsymbol{P}_R \overset{\text{def}}{=} \boldsymbol{G}_{Re} \tag{23}$$

is an invariant generalized 1-inverse of $\boldsymbol{\kappa}$. Hence, the special particular solution given by

$$\boldsymbol{C}_{R,\tau} = \boldsymbol{G}_{Re}\boldsymbol{F}_1 \tag{24}$$

is uniquely determined and falls into the complement space of the kernel of $\boldsymbol{\kappa}$. Then noticing definitions (18) and (21–23), we have

$$\boldsymbol{C}_{R,\tau} = -\boldsymbol{P}_L^T\boldsymbol{\kappa}^{\{1\}}\boldsymbol{P}_R\boldsymbol{\kappa}_\tau\boldsymbol{Z}_R \tag{25}$$

Furthermore,

$$\boldsymbol{Z}_{R,\tau} = \boldsymbol{C}_{R,\tau} + \boldsymbol{Z}_R\boldsymbol{D}_R \tag{26}$$

Next the coefficient matrix $\boldsymbol{D}_R$ in the homogeneous solution part should be found. By differentiating Eq. (13) along direction $\boldsymbol{\tau}$ and using Eq. (26), we get

$$\boldsymbol{D}_R\Lambda_{,\tau} - \Lambda_{,\tau}\boldsymbol{D}_R + 0.5\Lambda_{,\tau\tau} = \boldsymbol{Z}_L^T\boldsymbol{H}_R\boldsymbol{Z}_R + \boldsymbol{L}_R\Lambda_{,\tau} \overset{\text{def}}{=} \boldsymbol{U}_R \tag{27}$$

where

$$\boldsymbol{H}_R = 0.5\boldsymbol{\kappa}_{\tau\tau} - \boldsymbol{\kappa}_\tau\boldsymbol{G}_{Re}\boldsymbol{\kappa}_\tau \tag{28}$$

$$\boldsymbol{L}_R = -\boldsymbol{Z}_L^T\boldsymbol{M}_{,\tau}\boldsymbol{Z}_R \tag{29}$$

This provides a neat separation of matrices $\boldsymbol{D}_R$ and $\Lambda_{,\tau\tau}$ and allows solving for both the off-diagonal elements of matrix $\boldsymbol{D}_R$ and the second-order derivatives of the repeated eigenvalues $\Lambda_{,\tau\tau}$, although only $\boldsymbol{D}_R$ is of interest. If $\Lambda_{,\tau} = \text{diag}(\lambda_{,\tau}^1, \lambda_{,\tau}^2, \ldots, \lambda_{,\tau}^h)$ (here $\lambda_{,\tau}^i$ are all distinct, $i = 1, 2, \ldots, h$), $\Lambda_{,\tau\tau} = \text{diag}(\lambda_{,\tau\tau}^1, \lambda_{,\tau\tau}^2, \ldots, \lambda_{,\tau\tau}^h)$, $\boldsymbol{U}_R = [u_R^{ij}]$, and $\boldsymbol{D}_R = [d_R^{ij}]$, it is easily seen that

$$\lambda_{,\tau\tau}^i = 2u_R^{ii} \qquad (i = 1, 2, \ldots, h) \tag{30}$$

$$d_R^{ij} = u_R^{ij} \Big/ \left(\lambda_{,\tau}^j - \lambda_{,\tau}^i\right) \qquad (i \neq j;\ i, j = 1, 2, \ldots, h) \tag{31}$$

The diagonal elements of matrix $\boldsymbol{D}_R$ will be solved by using the normalization condition (10). Differentiating Eq. (10) along direction $\boldsymbol{\tau}$ and letting $\boldsymbol{d}_{Ri}$ and $\boldsymbol{c}_{Ri,\tau}$ denote ith column vectors of matrices $\boldsymbol{D}_R$ and $\boldsymbol{C}_{R,\tau}$ results in

$$z_{Ri}^T(\boldsymbol{M} + \boldsymbol{M}^T)\boldsymbol{Z}_R\boldsymbol{d}_{Ri} = -z_{Ri}^T\boldsymbol{M}_{,\tau}z_{Ri} - z_{Ri}^T(\boldsymbol{M} + \boldsymbol{M}^T)\boldsymbol{c}_{Ri,\tau} \overset{\text{def}}{=} 2\sigma_{ii} \tag{32}$$

Recalling Eq. (31), we can solve the diagonal elements of matrix $\boldsymbol{D}_R$

$$d_R^{ii} = \sigma_{ii} - \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{h} z_{Ri}^T(\boldsymbol{M} + \boldsymbol{M}^T)z_{Rj}d_R^{ji} \qquad (i = 1, 2, \ldots, h) \tag{33}$$

To obtain $\boldsymbol{Z}_{L,\tau}$ we only need to replace the subscript R with L and let $\boldsymbol{M}^T, \boldsymbol{K}^T$ represent $\boldsymbol{M}, \boldsymbol{K}$ in the preceding algorithm. It is then easy to obtain

$$\boldsymbol{Z}_{L,\tau} = \boldsymbol{C}_{L,\tau} + \boldsymbol{Z}_L\boldsymbol{D}_L \tag{34}$$

where $\boldsymbol{C}_{L,\tau} = -\boldsymbol{P}_R^T\boldsymbol{\kappa}^{\{1\}^T}\boldsymbol{P}_L\boldsymbol{\kappa}_\tau^T\boldsymbol{Z}_L$.

Differentiating Eq. (9) along direction $\boldsymbol{\tau}$ and noting expressions (26) and (34) yields

$$\boldsymbol{D}_L^T = -\boldsymbol{D}_R + \boldsymbol{Z}_L^T(\boldsymbol{\kappa}_\tau\boldsymbol{G}_{Re}\boldsymbol{M} + \boldsymbol{M}\boldsymbol{G}_{Re}\boldsymbol{\kappa}_\tau - \boldsymbol{M}_{,\tau})\boldsymbol{Z}_R \tag{35}$$

This completes the solution for $\boldsymbol{Z}_{t,\tau}$ $(t = R, L)$ and $\Lambda_{,\tau}$. Thereafter, evaluating

$$\boldsymbol{\tau} = \boldsymbol{e}_i = (0, \ldots, \underset{\substack{\Uparrow \\ i\text{th}}}{1}, \ldots, 0)^T, \qquad i = 1, 2, \ldots, n$$

which is the ith natural base vector of the n-dimensional parametric space in Eqs. (11a), (11b), (14), and (26), will lead to the partial derivatives of the repeated eigenvalue and the corresponding eigenvectors with respect to p_i:

$$\boldsymbol{Z}_t(\boldsymbol{e}_i) = \boldsymbol{X}_t \cdot \Gamma_t(\boldsymbol{e}_i, \boldsymbol{X}) \qquad (t = R, L) \qquad (i = 1, 2, \ldots, n) \tag{36}$$

$$\Lambda_{,e_i} = \boldsymbol{Z}_L(\boldsymbol{e}_i)^T \cdot \boldsymbol{\kappa}_{e_i} \cdot \boldsymbol{Z}_R(\boldsymbol{e}_i) \tag{37}$$

$$\boldsymbol{Z}_{R,e_i} = -\boldsymbol{P}_L^T\boldsymbol{\kappa}^{\{1\}}\boldsymbol{P}_R \cdot \boldsymbol{\kappa}_{e_i} \cdot \boldsymbol{Z}_R(\boldsymbol{e}_i) + \boldsymbol{Z}_R(\boldsymbol{e}_i) \cdot \boldsymbol{D}_R(\boldsymbol{e}_i) \tag{38}$$

Let matrix $\Theta_t(\boldsymbol{\tau}, \boldsymbol{e}_i)$, $t = R$ and L, be the directional transform matrices of eigenvectors evaluated at $\boldsymbol{p} = \boldsymbol{p}_0$ from direction $\boldsymbol{\tau}$ to a natural base $\boldsymbol{e}_i$, that is,

$$\boldsymbol{Z}_R(\boldsymbol{e}_i) = \boldsymbol{Z}_R(\boldsymbol{\tau}) \cdot \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) \tag{39}$$

$$\boldsymbol{Z}_L(\boldsymbol{e}_i) = \boldsymbol{Z}_L(\boldsymbol{\tau}) \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i) \tag{40}$$

the following restrictions should be imposed by Eqs. (36), (39) and (40):

$$\Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) = \Gamma_L(\boldsymbol{\tau}, \boldsymbol{X})^T \cdot \Gamma_R(\boldsymbol{e}_i, \boldsymbol{X}) \tag{41}$$

$$\Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i) = \Gamma_R(\boldsymbol{\tau}, \boldsymbol{X})^T \cdot \Gamma_L(\boldsymbol{e}_i, \boldsymbol{X}) \tag{42}$$

$$\Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) = \boldsymbol{I}_h \tag{43}$$

By virtue of the characteristic of stiffness matrix $\boldsymbol{K}$ and mass matrix $\boldsymbol{M}$, which have continuous partial derivatives of all orders with respect to multiple design parameters $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)^T$, matrix $\boldsymbol{\kappa} = \boldsymbol{K} - \lambda\boldsymbol{M}$ should be differentiable at $\boldsymbol{p} = \boldsymbol{p}_0$. Accordingly, the directional derivative can represented by

$$\boldsymbol{\kappa}_\tau = \sum_{i=1}^{n} \boldsymbol{\kappa}_{e_i} \cos\alpha_{\tau e_i} \tag{44}$$

where $\alpha_{\tau e_i}$ is the angle between the vector $\boldsymbol{\tau}$ and the natural base $\boldsymbol{e}_i$, $i = 1, 2, \ldots, n$. Substituting Eq. (14) with expressions (37), (39), (40), and (44) yields

$$\begin{aligned}\Lambda_{,\tau} &= \boldsymbol{Z}_L(\boldsymbol{\tau})^T \cdot \boldsymbol{\kappa}_\tau \cdot \boldsymbol{Z}_R(\boldsymbol{\tau}) \\ &= \sum_{i=1}^{n} \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) \cdot \boldsymbol{Z}_L(\boldsymbol{e}_i)^T \cdot \boldsymbol{\kappa}_{e_i} \cdot \boldsymbol{Z}_R(\boldsymbol{e}_i) \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \cos\alpha_{\tau e_i} \\ &= \sum_{i=1}^{n} \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) \cdot \Lambda_{,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \cos\alpha_{\tau e_i}\end{aligned} \tag{45}$$

It is well known that if an arbitrary multivariable function $\boldsymbol{F}(p_1, p_2, \ldots, p_n)$ is differentiable at $\boldsymbol{p} = \boldsymbol{p}_0$, not only the directional derivative $\boldsymbol{F}_{,\tau}$ exists but also it can be represented by

$$\boldsymbol{F}_{,\tau} = \left.\frac{\partial \boldsymbol{F}(\boldsymbol{p})}{\partial \boldsymbol{\tau}}\right|_{p=p_0} = \lim_{\substack{\Delta\tau\to 0 \\ \text{along }\tau}} \frac{\boldsymbol{F}(\boldsymbol{p}) - \boldsymbol{F}(\boldsymbol{p}_0)}{\Delta\tau} = \sum_{i=1}^{n} \boldsymbol{F}_{,e_i} \cos\alpha_{\tau e_i} \tag{46}$$

where

$$\Delta\tau = |\Delta\boldsymbol{p}| = \sqrt{\sum_{i=1}^{n} (\Delta p_i)^2} \tag{47}$$

However, in general $\Theta_t(\boldsymbol{\tau}, \boldsymbol{e}_i) \not\equiv \boldsymbol{I}_h, \forall\boldsymbol{\tau} \in R^n$, in expression (45) results in the repeated eigenvalue $\Lambda(\boldsymbol{p}) = \lambda\boldsymbol{I}_h$ not being differentiable at $\boldsymbol{p} = \boldsymbol{p}_0$. Similarly, in the case of the eigenvectors $\boldsymbol{Z}_R$, substituting Eq. (26) with expression (44), we get

$$\begin{aligned}\boldsymbol{Z}_{R,\tau} &= -\sum_{i=1}^{n} \boldsymbol{P}_L^T\boldsymbol{\kappa}^{\{1\}}\boldsymbol{P}_R \cdot \boldsymbol{\kappa}_{e_i} \cdot \boldsymbol{Z}_R(\boldsymbol{e}_i) \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cos\alpha_{\tau e_i} \\ &\quad + \boldsymbol{Z}_R(\boldsymbol{\tau}) \cdot \boldsymbol{D}_R(\boldsymbol{\tau}) \\ &= \sum_{i=1}^{n} \boldsymbol{C}_{R,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cos\alpha_{\tau e_i} + \boldsymbol{Z}_R(\boldsymbol{\tau}) \cdot \boldsymbol{D}_R(\boldsymbol{\tau})\end{aligned} \tag{48}$$

if and only if

$$\boldsymbol{D}_R(\boldsymbol{\tau}) = \sum_{i=1}^{n} \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) \cdot \boldsymbol{D}_R(\boldsymbol{e}_i) \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \cos\alpha_{\tau e_i} \tag{49}$$

we have

$$\boldsymbol{Z}_{R,\tau} = \sum_{i=1}^{n} \boldsymbol{C}_{R,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cos\alpha_{\tau e_i} + \sum_{i=1}^{n} \boldsymbol{Z}_R(\boldsymbol{e}_i) \cdot \boldsymbol{D}_R(\boldsymbol{e}_i) \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cos\alpha_{\tau e_i} = \sum_{i=1}^{n} \boldsymbol{Z}_{R,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cos\alpha_{\tau e_i} \tag{50}$$

Even if it possesses condition (49), the eigenvectors $\boldsymbol{Z}_R$ associated with the repeated eigenvalue are also not differentiable at $\boldsymbol{p} = \boldsymbol{p}_0$ due to $\Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i) \not\equiv \boldsymbol{I}_h$, $\forall \boldsymbol{\tau} \in R^n$, to say nothing of condition (49) not holding in general. It is the same as the left eigenvectors $\boldsymbol{Z}_L$.

Therefore, we can conclude that repeated eigenvalues and the corresponding eigenvectors of asymmetric multivariable systems are not differentiable.

Inverse Sensitivity Method

The aim of the inverse sensitivity method for parametric correction is to find the unknown change $\Delta\boldsymbol{p}$ of the physical parameters that is the difference between the experimental model and the analytical one by using the measured eigenquantities and the corresponding analytical representations. One takes into account only the first-order terms of the Taylor expansion in most references.

Suppose that $\bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m)$ and $\bar{\boldsymbol{Z}} = \{\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m\}$ are the first m orders test eigenvalues of the experimental model and the corresponding measured eigenvectors, respectively. Our task is to solve the physical parameters $\bar{\boldsymbol{p}} = \boldsymbol{p}_0 + \Delta\boldsymbol{p}$ in the experimental model. Here we have $\Lambda(\boldsymbol{p}) = \text{diag}[\lambda_1(\boldsymbol{p}), \lambda_2(\boldsymbol{p}), \ldots, \lambda_m(\boldsymbol{p})]$ and $\boldsymbol{Z}_R(\boldsymbol{p}) = \{z_{R1}(\boldsymbol{p}), z_{R2}(\boldsymbol{p}), \ldots, z_{Rm}(\boldsymbol{p})\}$ showing the first m-orders eigensolutions of the analytical model. The difference between $(\Lambda, \bar{\boldsymbol{Z}})$ and $(\Lambda, \boldsymbol{Z}_R)$ exists (hereafter $\boldsymbol{p}_0$ will be omitted for variables evaluated at $\boldsymbol{p} = \boldsymbol{p}_0$ as before) in general. We can regard $(\bar{\Lambda}, \bar{\boldsymbol{Z}})$ as the value of $[\Lambda(\boldsymbol{p}), \boldsymbol{Z}_R(\boldsymbol{p})]$ evaluated at $\boldsymbol{p} = \bar{\boldsymbol{p}}$. Hence, in terms of the first-order Taylor expansion of the eigenpairs along direction $\boldsymbol{\tau}$ at $\boldsymbol{p} = \boldsymbol{p}_0$, we have

$$\bar{\Lambda} - \Lambda = \Lambda_{,\tau} \cdot \Delta\tau \tag{51}$$

$$\bar{\boldsymbol{Z}} - \boldsymbol{Z}_R = \boldsymbol{Z}_{R,\tau} \cdot \Delta\tau \tag{52}$$

Denoting $\Delta\boldsymbol{w} = [(\bar{\lambda}_1 - \lambda_1), (\bar{\lambda}_2 - \lambda_2), \ldots, (\bar{\lambda}_m - \lambda_m), (\bar{z}_1 - z_{R1})^T, (\bar{z}_2 - z_{R2})^T, \ldots, (\bar{z}_m - z_{Rm})^T]^T$ and $\Delta\boldsymbol{p} = (\Delta p_1, \Delta p_2, \ldots, \Delta p_n)^T$, we introduce the following sensitivity equation:

$$\Delta\boldsymbol{w} = \boldsymbol{L}_{m(N+1)\times n} \cdot \Delta\boldsymbol{p} \tag{53}$$

In the case of simple eigenvalues, because the eigenpairs are differentiable and noting Eq. (46), Eqs. (51) and (52) can be expressed as

$$\bar{\Lambda} - \Lambda = \Lambda_{,\tau} \cdot \Delta\tau = \sum_{i=1}^{n} \Lambda_{,e_i} \cos\alpha_{\tau e_i} \cdot \Delta\tau = \sum_{i=1}^{n} \Lambda_{,e_i} \cdot \Delta p_i \tag{54}$$

$$\bar{\boldsymbol{Z}} - \boldsymbol{Z}_R = \boldsymbol{Z}_{R,\tau} \cdot \Delta\tau = \sum_{i=1}^{n} \boldsymbol{Z}_{R,e_i} \cos\alpha_{\tau e_i} \cdot \Delta\tau = \sum_{i=1}^{n} \boldsymbol{Z}_{R,e_i} \cdot \Delta p_i \tag{55}$$

Hence, the operator $\boldsymbol{L}$ in Eq. (53) is easily obtained, namely, the Jacobian matrix.

However, difficulty arises when repeated eigenvalues occur in the analytical model due to the nondifferentiability of the repeated root eigenpairs. Here the case of an arbitrary h-fold $(1 < h \le N)$ eigenvalue $\Lambda(\boldsymbol{p}) = \lambda\boldsymbol{I}_h$ with all distinct derivatives is considered. Recalling expression (45) and substituting it into Eq. (51), we find

$$\bar{\Lambda} - \Lambda = \sum_{i=1}^{n} \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) \cdot \Lambda_{,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cos\alpha_{\tau e_i} \cdot \Delta\tau = \sum_{i=1}^{n} \Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) \cdot \Lambda_{,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \Delta p \tag{56}$$

Denoting $\Delta\lambda = [(\bar{\lambda}_1 - \lambda), (\bar{\lambda}_2 - \lambda), \ldots, (\bar{\lambda}_h - \lambda)]^T$, if $\Theta_R(\boldsymbol{\tau}, \boldsymbol{e}_i) = [\theta_R^{ij}]$ and $\Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i) = [\theta_L^{ij}]$, expression (56) can be rewritten into a matrix form

$$\Delta\lambda = \boldsymbol{J}_\lambda \cdot \Delta\boldsymbol{p} \tag{57}$$

where

$$\boldsymbol{J}_\lambda = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{h1} & t_{h2} & \cdots & t_{hn} \end{bmatrix} \tag{58}$$

$$t_{jk} = \sum_{s=1}^{h} \theta_R^{js} \cdot \lambda_{,e_k}^{s} \cdot \theta_L^{js} \qquad (j = 1, 2, \ldots, h;\ k = 1, 2, \ldots, n) \tag{59}$$

Then we take the advantage of the projection operator $\boldsymbol{P}_L$, which has the following characteristic:

$$\boldsymbol{P}_L^T \boldsymbol{Z}_R = \boldsymbol{0} \tag{60}$$

$$\boldsymbol{P}_L^T \boldsymbol{C}_{R,\tau} = \boldsymbol{C}_{R,\tau} \tag{61}$$

Left-multiplying equation (52) by $\boldsymbol{P}_L^T$ in order to reject the homogeneous solution part and by using (60) and (61), we have

$$\boldsymbol{P}_L^T \bar{\boldsymbol{Z}} = \boldsymbol{C}_{R,\tau} \cdot \Delta\tau = \sum_{i=1}^{n} \boldsymbol{C}_{R,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \cos\alpha_{\tau e_i} \cdot \Delta\tau = \sum_{i=1}^{n} \boldsymbol{C}_{R,e_i} \cdot \Theta_L(\boldsymbol{\tau}, \boldsymbol{e}_i)^T \cdot \Delta p_i \tag{62}$$

Rewriting expression (62) into a matrix form

$$\Delta\bar{z} = \boldsymbol{J}_z \cdot \Delta\boldsymbol{p} \tag{63}$$

where

$$\Delta\bar{z} = \begin{bmatrix} \boldsymbol{P}_L^T \bar{z}_1 \\ \boldsymbol{P}_L^T \bar{z}_2 \\ \vdots \\ \boldsymbol{P}_L^T \bar{z}_h \end{bmatrix}, \qquad \boldsymbol{J}_z = \begin{bmatrix} \boldsymbol{v}_R^{11} & \boldsymbol{v}_R^{12} & \cdots & \boldsymbol{v}_R^{1n} \\ \boldsymbol{v}_R^{21} & \boldsymbol{v}_R^{22} & \cdots & \boldsymbol{v}_R^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_R^{h1} & \boldsymbol{v}_R^{h1} & \cdots & \boldsymbol{v}_R^{hn} \end{bmatrix} \tag{64}$$

$$\boldsymbol{v}_R^{jk} = \sum_{s=1}^{h} \boldsymbol{c}_{Rs,e_k} \cdot \theta_L^{js} \qquad (j = 1, 2, \ldots, h;\ k = 1, 2, \ldots, n) \tag{65}$$

Synthetically, if $\Delta\boldsymbol{E} = (\Delta\lambda, \Delta\bar{z})^T$ and $\boldsymbol{J}_e = (\boldsymbol{J}_\lambda, \boldsymbol{J}_z)^T$ is the so-called generalized Jacobian matrix, we have the sensitivity equation of the repeated eigenvalues systems

$$\Delta\boldsymbol{E} = \boldsymbol{J}_e \cdot \Delta\boldsymbol{p} \tag{66}$$

then an unknown vector $\Delta\boldsymbol{p}$ is expressed by the inverse sensitivity equation

$$\Delta\boldsymbol{p} = \left[\boldsymbol{J}_e^T \boldsymbol{J}_e\right]^{-1} \cdot \boldsymbol{J}_e^T \cdot \Delta\boldsymbol{E} = \boldsymbol{J}_e^{+} \cdot \Delta\boldsymbol{E} \tag{67}$$

Finally we should determine the $h \times h$ transform matrices Γ_t, $t = R$ and L to get the directional transform matrices $\Theta_t(\boldsymbol{\tau}, \boldsymbol{e}_i)$ and the derivable eigenvectors $\boldsymbol{Z}_t$. It can be assumed that the eigenvectors from $\boldsymbol{Z}_R = \boldsymbol{X}_R \Gamma_R$ very well approximate the eigenvectors from $\bar{\boldsymbol{Z}}$ because $\Delta\tau$ is so small. Then Γ_R can be found by minimizing the trace of the matrix $(\boldsymbol{X}_R\Gamma_R - \bar{\boldsymbol{Z}})^T (\boldsymbol{X}_R\Gamma_R - \bar{\boldsymbol{Z}})$ in the sense of least squares[5] and be expressed as

$$\Gamma_R = \left(\boldsymbol{X}_R^T \boldsymbol{X}_R\right)^{-1} \left(\boldsymbol{X}_R^T \bar{\boldsymbol{Z}}\right) \tag{68}$$

By condition (17), Γ_L can be determined accordingly.

Illustrative Examples

Two numerical examples are given to demonstrate the availability and efficiency of the presented method by using simulated data. All results are obtained by using MATHEMATICA 3.0.

Example 1

Suppose that an eight-degree-of-freedom discrete vibration system $[\boldsymbol{K}(\boldsymbol{p}), \boldsymbol{M}(\boldsymbol{p})]$ is given by

$$\boldsymbol{K}(\boldsymbol{p}) = \begin{bmatrix} 1+p_3 & p_1+p_2 & 0 & 0 & -1 & 1 & 0 & 1 \\ p_2-p_1 & 1+p_4 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & p_1+p_2 & -1 & 1 & 0 & 1 \\ 0 & 0 & p_2-p_1 & 10 & -1 & 5 & 0 & 0 \\ 2 & 6 & 3 & 4 & 10 & 3+p_1+p_2 & 0 & 10 \\ 1 & 1 & 1 & 1 & 1+p_2-p_1 & 10 & 0 & 10 \\ 20 & 0 & 0 & 0 & 0 & 0 & 5 & 20+p_1+p_2 \\ 3 & 7 & 4 & 5 & 11 & 13 & p_2-p_1 & 20 \end{bmatrix}, \qquad \boldsymbol{M}(\boldsymbol{p}) = \boldsymbol{I}_8$$

where $\boldsymbol{p} = (p_1, p_2, p_3, p_4)^T$ are the design parameters, and at $\boldsymbol{p} = \boldsymbol{p}_0 = (0, 0, 0, 0)^T$, the original system has a set of the complete eigenvalue

$$\boldsymbol{\Omega} = \text{diag}(0.0 \quad 1.0 \quad 1.0 \quad 1.60289 \quad 5.0 \quad 8.32103 - 2.54373I \quad 8.32103 + 2.54373I \quad 32.755), \qquad I = \sqrt{-1}$$

with a twofold eigenvalue $\lambda_2 = \lambda_3 = \lambda = 1.0$, whose corresponding biorthonormalized right and left eigenvectors are, respectively,

$$\boldsymbol{X}_R = \begin{bmatrix} -1.10342 & -1.25619 \\ 0.463194 & 0.443361 \\ -1.35418 & -1.25619 \\ 0.019944 & 0.0206902 \\ 0.179496 & 0.186212 \\ 0 & 0 \\ 4.61961 & 5.34989 \\ 0.179496 & 0.186212 \end{bmatrix}$$

and the transform matrices $\Gamma_t(\boldsymbol{e}_i, \boldsymbol{X})$ $t = R, L$, with respect to parameter p_i. In virtue of definitions (21) and (22), the projection matrices of the repeated eigenvalue are

$$\boldsymbol{P}_R = \boldsymbol{P}_L^T = \begin{bmatrix} 0.0434783 & 0.73913 & 0.217391 & 0 & 0 & 0 & 0 & 0 \\ 0.0434783 & 0.73913 & 0.217391 & 0 & 0 & 0 & 0 & 0 \\ 0.0434783 & 0.73913 & 0.217391 & 0 & 0 & 0 & 0 & 0 \\ 0.00869565 & -0.0121739 & 0.00347826 & 1 & 0 & 0 & 0 & 0 \\ 0.0782609 & -0.109565 & 0.0313043 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4.3913 & -3.14783 & -1.24348 & 0 & 0 & 0 & 1 & 0 \\ 0.0782609 & -0.109565 & 0.0313043 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $\boldsymbol{G}_{Re}$, the invariant generalized 1-inverse of $\boldsymbol{\kappa}$, can be calculated by definition (23); accordingly,

$$\boldsymbol{G}_{Re} = \begin{bmatrix} 0.094518 & 1.36333 & 0.41172 & -0.0434783 & -0.391304 & -0.869565 & 0 & 0.565217 \\ 0.094518 & 1.36333 & 0.41172 & -0.0434783 & -0.391304 & -0.869565 & 0 & 0.565217 \\ 0.094518 & 1.36333 & 0.41172 & -0.0434783 & -0.391304 & -0.869565 & 0 & 0.565217 \\ -0.017966 & -0.315717 & -0.0924038 & 0.111304 & 0.281739 & 0.306087 & 0 & -0.286957 \\ -0.0616938 & -0.981455 & -0.291634 & 0.00173913 & 0.0356522 & 0.254783 & 0 & -0.0826087 \\ 0.0217391 & 0.369565 & 0.108696 & 0 & -0.5 & -0.5 & 0 & 0.5 \\ 0.825009 & -4.54414 & -1.45478 & 0.208696 & -0.721739 & 0.573913 & 0.25 & 0.0869565 \\ -0.0399546 & -0.61189 & -0.182938 & 0.00173913 & 0.535652 & 0.754783 & 0 & -0.582609 \end{bmatrix}$$

$$\boldsymbol{X}_L = \begin{bmatrix} 3.98791 & -4.26436 \\ 0 & 0.588391 \\ -3.98791 & 3.67597 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\lambda_1 = 0.0$ as a simple eigenvalue, both the eigenvalue and the corresponding eigenvectors $\boldsymbol{y}_t$, $t = R$ and L are differentiable. Here we use the first three eigenvalues including a simple one and a twofold one to correct the parameters. Solving the eigenproblems (15–17) evaluated at $\boldsymbol{\tau} = \boldsymbol{e}_i$, $i = 1, 2, 3, 4$, respectively, we have the partial derivatives of the repeated eigenvalue

$$\Lambda_{,e_1} = \begin{bmatrix} 0.346087 + 0.301096I & 0 \\ 0 & 0.346087 - 0.301096I \end{bmatrix}$$

$$\Lambda_{,e_2} = \begin{bmatrix} -1 & 0 \\ 0 & 0.213913 \end{bmatrix}, \qquad \Lambda_{,e_3} = \begin{bmatrix} 0.956522 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Lambda_{,e_4} = \begin{bmatrix} 0.26087 & 0 \\ 0 & 0 \end{bmatrix}$$

Thereafter the special particular solutions $\boldsymbol{C}_{R,e_1}$, $i = 1, 2, 3, 4$, can be solved, respectively:

$$\boldsymbol{C}_{R,e_1} = \begin{bmatrix} 0.200496 + 0.0552401I & 0.200496 - 0.0552401I \\ 0.200496 + 0.0552401I & 0.200496 - 0.0552401I \\ 0.200496 + 0.0552401I & 0.200496 - 0.0552401I \\ -0.221717 - 0.00157521I & -0.221717 + 0.00157521I \\ 0.162299 - 0.0148762I & 0.162299 + 0.0148762I \\ 0.376137 + 0.0494534I & 0.376137 - 0.0494534I \\ 1.01071 + 0.112266I & 1.01071 - 0.112266I \\ -0.389008 - 0.0610037I & -0.389008 + 0.0610037I \end{bmatrix}$$

$$\boldsymbol{C}_{R,e_2} = \begin{bmatrix} -0.229769 & 0.251493 \\ -0.229769 & 0.251493 \\ -0.229769 & 0.251493 \\ 0.216057 & -0.137466 \\ -0.13994 & 0.0788748 \\ -0.384918 & 0.36754 \\ -1.1223 & 0.705339 \\ 0.404839 & -0.402459 \end{bmatrix}$$

$$\boldsymbol{C}_{R,e_3} = \begin{bmatrix} -0.0201084 & 0 \\ -0.0201084 & 0 \\ -0.0201084 & 0 \\ 0.0038222 & 0 \\ 0.0131251 & 0 \\ -0.00462492 & 0 \\ -0.175518 & 0 \\ 0.00850021 & 0 \end{bmatrix}, \qquad \boldsymbol{C}_{R,e_4} = \begin{bmatrix} 0.106779 & 0 \\ 0.106779 & 0 \\ 0.106779 & 0 \\ -0.0247277 & 0 \\ -0.0768699 & 0 \\ 0.0289453 & 0 \\ -0.355908 & 0 \\ -0.0479247 & 0 \end{bmatrix}$$

The other mechanical system $[\boldsymbol{K}(\bar{\boldsymbol{p}}), \boldsymbol{M}(\boldsymbol{p})]$ to be considered, called the target system, is distinguished from the original analytical system by changing the physical parameters, $\bar{\boldsymbol{p}} = \boldsymbol{p}_0 + \Delta\boldsymbol{p}$. The aim is to evaluate the unknown vector $\Delta\boldsymbol{p} = (\Delta\boldsymbol{p}_1, \Delta\boldsymbol{p}_2, \Delta\boldsymbol{p}_3, \Delta\boldsymbol{p}_4)^T$ by means of the earlier described procedure. With the assumption that, for case 1.1, $\bar{\boldsymbol{p}} = (-0.2, 0.25, 0.3, -0.15)^T$, the first three eigenvalues and the corresponding biorthonormalized eigenvectors of the target system are

$$\bar{\Lambda} = \begin{bmatrix} -0.368132 & 0 & 0 \\ 0 & 0.826429 & 0 \\ 0 & 0 & 1.12255 \end{bmatrix}$$

$$\bar{\boldsymbol{Z}} = \begin{bmatrix} -0.126099 & 0.274529 & -0.281832 \\ -0.131496 & 0.231974 & 0.251351 \\ -0.163325 & -0.510083 & -0.721273 \\ 0.130522 & -0.0440098 & 0.0858417 \\ -0.154866 & 0.0703723 & -0.0549059 \\ -0.286929 & 0.138121 & -0.103562 \\ -0.833662 & -0.753838 & 0.536005 \\ 0.348987 & -0.128061 & 0.169556 \end{bmatrix}$$

When Eq. (68) and expressions (11a) and (11b) are used, the transform matrices $\Gamma_t(\boldsymbol{\tau}, \boldsymbol{X})$, $t = R$ and L, for the repeated eigenvalue and the derivable biorthonormalized eigenvectors with respect to the first three eigenvalues of the original system can be obtained:

$$\Gamma_R(\boldsymbol{\tau}, \boldsymbol{X}) = \begin{bmatrix} 2.71284 & 2.30883 \\ -2.49502 & -1.88596 \end{bmatrix}$$

$$\Gamma_L(\boldsymbol{\tau}, \boldsymbol{X}) = \begin{bmatrix} -2.92739 & 3.87277 \\ -3.58376 & 4.21087 \end{bmatrix}$$

$$\Phi_R = [\boldsymbol{y}_R \quad \boldsymbol{Z}_R] = \begin{bmatrix} -0.191228 & 0.14082 & -0.178473 \\ -0.191228 & 0.150378 & 0.233272 \\ -0.191228 & -0.539449 & -0.75743 \\ 0.150509 & 0.00248251 & 0.00702631 \\ -0.135373 & 0.0223426 & 0.0632368 \\ -0.328093 & 0 & 0 \\ -0.77088 & -0.815811 & 0.57618 \\ 0.383948 & 0.0223426 & 0.0632368 \end{bmatrix}$$

Then the directional transform matrices $\Theta_t(\boldsymbol{\tau}, \boldsymbol{e}_i)$, $t = R$ and L, can be determined by Eqs. (41) and (42).

Thus, by the use of Eqs. (58–67), $\Delta\boldsymbol{E}$ and the generalized Jacobian matrix $\boldsymbol{J}_e$ have been formed. Then the correction of the parameters $\Delta\boldsymbol{p}$ is found to be

$$\Delta\boldsymbol{p} = (-0.230659, 0.274681, 0.268452, -0.134342)^T$$

Solving the generalized eigenproblems (1) and (2) at

$$\boldsymbol{p} = \boldsymbol{p}^* = \boldsymbol{p}_0 + \Delta\boldsymbol{p}$$

$$= (-0.230659, 0.274681, 0.268452, -0.134342)^T$$

we can find that the eigenvalues are all distinct. Then another correction for parameters $\boldsymbol{p}$ evaluated at $\boldsymbol{p} = \boldsymbol{p}^*$ by Eqs. (54) and (55) results in a set of new parameters,

$$\boldsymbol{p}^{**} = \boldsymbol{p}^* + \Delta\boldsymbol{p}^* = (-0.19862, 0.254227, 0.30175, -0.146768)^T$$

which is very close to the exact data $\bar{\boldsymbol{p}} = (-0.2, 0.25, 0.3, -0.15)^T$. Table 1 lists the eigenvalues $\lambda(\boldsymbol{p})$ evaluated at $\boldsymbol{p} = \bar{\boldsymbol{p}}, \boldsymbol{p} = \boldsymbol{p}^{**}, \boldsymbol{p} = \boldsymbol{p}^*$, and $\boldsymbol{p} = \boldsymbol{p}_0$, respectively. The correlation between the eigenmodes $\boldsymbol{Z}_R(\boldsymbol{p})$ and the exact data $\bar{\boldsymbol{Z}}$ were evaluated by the modal assurance criterion (MAC).[13] Table 2 lists the MAC($\boldsymbol{p}$) evaluated at

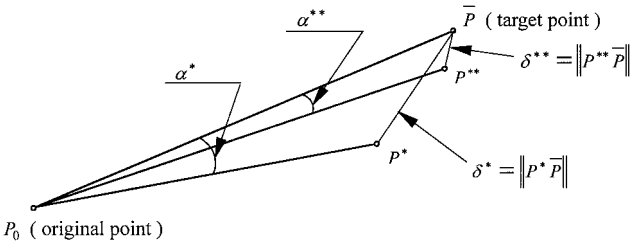

Fig. 1 Relation positions among P_0, P^*, P^{**}, and $\bar{P}$.

Table 1 Eigenvalue comparison for case 1.1[a]

Eigenvalue	$\lambda(\bar{\boldsymbol{p}})$	$\lambda(\boldsymbol{p}^{**})$	$\lambda(\boldsymbol{p}^*)$	$\lambda(\boldsymbol{p}_0)$
λ_8	33.075	33.0778	33.112	32.755
λ_7	8.14943+2.48275I	8.14696+2.48201I	8.12704+2.47371I	8.32103+2.54373I
λ_6	8.14943−2.48275I	8.14696−2.48201I	8.12704−2.47371I	8.32103−2.54373I
λ_5	5.46523	5.4861	5.52248	5.0
λ_4	1.73009	1.73	1.7756	1.60289
λ_3	1.12255	1.1246	1.0841	1.0
λ_2	0.826429	0.824903	0.787681	1.0
λ_1	−0.368132	−0.369789	−0.401809	0.0

[a]Case 1: $\alpha^* = 0.11314$ rad, $\delta^* = 0.113906 \cdot \|P_0\bar{P}\|$, approaches target point about 88.6%; and $\alpha^{**} = 0.0117962$ rad, $\delta^{**} = 0.0124423 \cdot \|P_0\bar{P}\|$, approaches target point about 98.8%.

Table 2 MAC($\boldsymbol{p}$) comparison for case 1.1

Eigenvector	MAC($\bar{\boldsymbol{p}}$)	MAC($\boldsymbol{p}^{**}$)	MAC($\boldsymbol{p}^*$)	MAC($\boldsymbol{p}_0$)
z_{R8}	1	1.0	0.999999	0.999967
z_{R7}	1	1.0	1.00005+0.0000357032I	1.00606+0.00189573I
z_{R6}	1	1.0	1.00005−0.0000357032I	1.00606−0.00189573I
z_{R5}	1	1.0	0.999942	0.996363
z_{R4}	1	0.99999	0.995626	0.970979
z_{R3}	1	0.999986	0.947671	0.926015
z_{R2}	1	0.999987	0.994812	0.944658
z_{R1}	1	0.999997	0.999801	0.98384

Table 3 Eigenvalue comparison for case 1.2[a]

Eigenvalue	$\lambda(\bar{\boldsymbol{p}})$	$\lambda(\boldsymbol{p}^{**})$	$\lambda(\boldsymbol{p}^{*})$	$\lambda(\boldsymbol{p}_0)$
λ_8	32.8779	32.8834	32.9261	32.755
λ_7	8.22773+2.52512I	8.22416+2.52412I	8.20753+2.51594I	8.32103+2.54373I
λ_6	8.22773−2.52512I	8.22416−2.52412I	8.20753−2.51594I	8.32103−2.54373I
λ_5	5.14123	5.14838	5.21118	5.0
λ_4	1.89656	1.91027	1.88042	1.60289
λ_3	1.03764	1.03836	1.03474	1.0
λ_2	0.662534	0.654471	0.559484	1.0
λ_1	−0.0213434	−0.0239516	−0.0564359	0.0

[a]Case 1.2: $\alpha^* = 0.223796$ rad, $\delta^* = 0.222208 \cdot \|P_0\bar{P}\|$, approaches target point about 77.8%; and $\alpha^{**} = 0.0171757$ rad, $\delta^{**} = 0.0315913 \cdot \|P_0\bar{P}\|$, approaches target point about 96.8%.

Table 4 MAC($\boldsymbol{p}$) comparison for case 1.2

Eigenvector	MAC($\bar{\boldsymbol{p}}$)	MAC($\boldsymbol{p}^{**}$)	MAC($\boldsymbol{p}^{*}$)	MAC($\boldsymbol{p}_0$)
z_{R8}	1	1.0	0.999999	0.999998
z_{R7}	1	1.0	1.00009+0.0I	0.999805−0.00036969I
z_{R6}	1	1.0	1.00009−0.0I	0.999805+0.00036969I
z_{R5}	1	0.999999	0.99991	0.999633
z_{R4}	1	0.999963	0.99862	0.939906
z_{R3}	1	0.999878	0.996372	0.99289
z_{R2}	1	0.999806	0.977399	0.964945
z_{R1}	1	0.999991	0.999439	0.985199

Table 5 Mode correction results for case 2.1

Parameter	$\Delta\boldsymbol{p} = \bar{\boldsymbol{p}} - \boldsymbol{p}_0 = (-0.05, 0.02)^T$, $\|\Delta\boldsymbol{p}\| = 0.0538516$			
Eigenvalue	$\lambda(\bar{\boldsymbol{p}})$	$\lambda(\boldsymbol{p}^{**})$	$\lambda(\boldsymbol{p}^{*})$	$\lambda(\boldsymbol{p}_0)$
λ_1	2.49375	2.49374	2.47686	3
λ_2	3.02837	3.02835	3.03024	3
λ_3	5.66629	5.66627	5.6538	6
Eigenvector	MAC($\bar{\boldsymbol{p}}$)	MAC($\boldsymbol{p}^{**}$)	MAC($\boldsymbol{p}^{*}$)	MAC($\boldsymbol{p}_0$)
z_{R1}	1	1	0.999978	0.998652
z_{R2}	1	1	1	0.999967
z_{R3}	1	1	0.999983	0.985061
α	0	0.000257543	0.0147019	1
$\delta/\|P_0\bar{P}\|$	0	0.000291533	0.0305396	1

Table 6 Mode correction results for case 2.2

Parameter	$\Delta\boldsymbol{p} = \bar{\boldsymbol{p}} - \boldsymbol{p}_0 = (0.1, -0.15)^T$, $\|\Delta\boldsymbol{p}\| = 0.180278$			
Eigenvalue	$\lambda(\bar{\boldsymbol{p}})$	$\lambda(\boldsymbol{p}^{**})$	$\lambda(\boldsymbol{p}^{*})$	$\lambda(\boldsymbol{p}_0)$
λ_1	2.81119	2.81232	2.84774	3
λ_2	3.84906	3.84746	3.76837	3
λ_3	6.52069	6.51941	6.46268	6
Eigenvector	MAC($\bar{\boldsymbol{p}}$)	MAC($\boldsymbol{p}^{**}$)	MAC($\boldsymbol{p}^{*}$)	MAC($\boldsymbol{p}_0$)
z_{R1}	1	1	0.999911	0.997895
z_{R2}	1	1	0.999741	0.987997
z_{R3}	1	1	0.999069	0.932734
α	0	0.0025985	0.0802445	1
$\delta/\|P_0\bar{P}\|$	0	0.00686904	0.219529	1

Table 7 Mode correction results for case 2.3

Parameter	$\Delta\boldsymbol{p} = \bar{\boldsymbol{p}} - \boldsymbol{p}_0 = (-0.08, -0.3)^T$, $\|\Delta\boldsymbol{p}\| = 0.310483$			
Eigenvalue	$\lambda(\bar{\boldsymbol{p}})$	$\lambda(\boldsymbol{p}^{**})$	$\lambda(\boldsymbol{p}^{*})$	$\lambda(\boldsymbol{p}_0)$
λ_1	2.18586	2.18565	2.1148	3
λ_2	3.34037	3.34106	3.35725	3
λ_3	5.49529	5.49539	5.44475	6
Eigenvector	MAC($\bar{\boldsymbol{p}}$)	MAC($\boldsymbol{p}^{**}$)	MAC($\boldsymbol{p}^{*}$)	MAC($\boldsymbol{p}_0$)
z_{R1}	1	1	0.999919	0.984441
z_{R2}	1	1	0.9999	0.967366
z_{R3}	1	1	0.999707	0.962966
α	0	0.000856806	0.0184984	1
$\delta/\|P_0\bar{P}\|$	0	0.00358589	0.0213743	1

$\boldsymbol{p} = \bar{\boldsymbol{p}}$, $\boldsymbol{p} = \boldsymbol{p}^{**}$, $\boldsymbol{p} = \boldsymbol{p}^{*}$, and $\boldsymbol{p} = \boldsymbol{p}_0$, respectively. Figure 1 shows the relative position among the original point P_0, first correction point P^*, second correction point P^{**}, and the target point $\bar{P}$. Note that the closer α and δ is to zero, the better is the correction for parameters.

Letting another arbitrary set of parameters $\bar{\boldsymbol{p}} = (0.01, 0.15, -0.2, 0.25)^T$ (case 1.2), we have the results shown in Tables 3 and 4.

Example 2

In this example, the structural mass matrices $\boldsymbol{M}(\boldsymbol{p})$ are not constant. Suppose that

$$\boldsymbol{M}(\boldsymbol{p}) = \begin{bmatrix} p^3+1 & 10 & 3 \\ q^2-1 & (1+p)^3+(1+p)^2+(1+p) & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\boldsymbol{K}(\boldsymbol{p}) = \begin{bmatrix} (1+p)^3+(1+p)^2+(1+p) & 38 & 17 \\ (1+q)^3 & (2+p)^3+2(2+p)^2+(2+p)+2 & 20 \\ 4 & 16 & (1+p)^3+2 \end{bmatrix}$$

where $\boldsymbol{p} = (p, q)^T$ are the design parameters, and at $\boldsymbol{p} = \boldsymbol{p}_0 = (1, 1)^T$, the original system has the following eigenvalues: $\boldsymbol{\Omega} = \mathrm{diag}(3, 3, 6)$, with a twofold eigenvalues $\lambda_1 = \lambda_2 = \lambda = 3$, whose corresponding biorthonormalized right and left eigenvectors are, respectively,

$$\boldsymbol{X}_R = \begin{bmatrix} -1.0 & -0.361039 \\ 0.0 & -0.268889 \\ 1.0 & 0.629928 \end{bmatrix}$$

$$\boldsymbol{X}_L = \begin{bmatrix} -0.333333 & 0.0 \\ 0.390451 & -0.619834 \\ -0.114235 & 1.23967 \end{bmatrix}$$

Cases 2.1–2.3 of three arbitrary sets of parameters $\bar{\boldsymbol{p}}$ are corrected. The results are shown in Tables 5–7. Obviously, after two corrections, the original point is very close to the target point.

Conclusions

An inverse sensitivity method for parametric correction is developed for asymmetric multivariable nondefective systems. It is proved that the repeated root eigenpairs are not differentiable for design parameters. This shows that the Jacobian matrix depends on the parameter change $\Delta \boldsymbol{p}$ in the sensitivity equation based on the first Taylor expansion. Therefore, the generalized Jacobian matrix has been presented so that the inverse sensitivity method for parametric correction can be carried out. The procedure has been shown for two numerical asymmetric examples by using simulated measured (exact) data, which can exclude measurement errors. The calculated results for two cases indicate that the original point changes converged very rapidly to the target point, which shows the applicability and efficiency of the method. The next-level derivatives of the eigenproblem are necessary to determine the transform matrix $\Gamma_t(\boldsymbol{e}_i, \boldsymbol{X})$, $i = 1, 2, \ldots, n$, with both repeated eigenvalues and repeated eigenvalue derivatives. Then the processes that follow are the same as the described method.

Acknowledgments

This work is a part of Projects 19972016 and 19472022 supported by National Natural Science Foundation of the People's Republic of China. The support is gratefully acknowledged.

References

[1]Nelson, U. R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[2]Dailey, R. L., "Eigenvector Derivatives with Repeated Eigenvalues," *AIAA Journal*, Vol. 27, No. 4, 1989, pp. 486–491.

[3]Mills-Curran, W. C., "Calculation of Eigenvector Derivatives for Structure with Repeated Eigenvalues," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 867–871.

[4]Prells, U., and Friswell, M. I., "Partial Derivatives of Repeated Eigenvalues and Their Eigenvectors," *AIAA Journal*, Vol. 35, No. 8, 1997, pp. 1363–1368.

[5]Pešěk, L., "An Extension of the Inverse Sensitivity Method to Systems with Repeated Eigenvalues," *Journal of Sound and Vibration*, Vol. 182, No. 4, 1995, pp. 623–635.

[6]Song, H. W., Chen, L. F., and Wang, W. L., "Inverse Sensitivity Method to Self-Adjoint Systems with Repeated Eigenvalues," *AIAA Journal*, Vol. 37, No. 8, 1999, pp. 933–938.

[7]Caughey, T. K., and Ma, F., "Complex Modes and Solvability of Nonclassical Systems," *Journal of Applied Mechanics*, Vol. 60, No. 3, 1993, pp. 26–28.

[8]Tang, J., and Wang, W. L., "On Calculation of Sensitivity for Non-Defective Eigenproblems with Repeated Roots," *Journal of Sound and Vibration*, Vol. 225, No. 4, 1999, pp. 611–631.

[9]Wang, W. L., "A Family of Modal Methods for Computing Eigenvector Derivatives with Repeated Roots," *Acta Mechanica Sinica* (English Series), Vol. 12, No. 2, 1996, pp. 158–168.

[10]Leongo, A., "Eigensolutions Sensitivity for Nonsymmetric Matrices with Repeated Eigenvalues," *AIAA Journal*, Vol. 31, No. 7, 1993, pp. 1321–1328.

[11]Craig, R. R., *Structural Dynamics—An Introduction to Computer Methods*, Wiley, New York, 1981.

[12]Murthy, D. V., and Haftka, R. T., "Derivatives of Eigenvalues and Eigenvectors of a General Complex Matrix," *International Journal for Numerical Methods in Engineering*, Vol. 16, No. 2, 1988, pp. 293–311.

[13]Imregun, M., and Wisser, W. J., "A Review of Modal Updating Techniques," *Shock and Vibration Digest*, Jan. 1991, pp. 9–20.

A. Berman
Associate Editor